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ON THE SOLVABILITY CONDITIONS OF THE BOUNDARY VALUE PROBLEMS FOR ONE CLASS OPERATOR-DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

Abstract

Sufficient conditions in terms of the coefficients of elliptic type operator-differential equation of the second order are obtained in the paper. These conditions provide solvability of some boundary value problems for this equation.

Let $H$ be a separable Hilbert space, $A$ is positively-defined self-adjoint operator in $H$. Denote by $H_y, (y \geq 0)$ the scale of Hilbert spaces, generated by the operator $A$, i.e. $H_y = D(A^y), (x, y) = (A^y x, A^y y)$.

Denote by $L_2(R_+; H)$ the set of measurable vector-functions $f(t)$ with values from $H$, for which

$$\|f\|_{L_2(R_+; H)} = \int_0^\infty |f(t)|^2 \, dt < \infty.$$ 

Introduce the following Hilbert spaces [1]:

$W^2_1(R_+; H) = \{u: A^2 u \in L_2(R_+; H), u'' \in L_2(R_+; H)\}$,

$W^2_2(R_+; H; 0) = \{u: u \in W^2_2(R_+; H), u(0) = 0\}$,

$W^2_2(R_+; H; 1) = \{u: u \in W^2_2(R_+; H), u'(0) = 0\}$,

$W^2_2(R_+; H; 0; 1) = \{u: u \in W^2_2(R_+; H), u(0) = 0, u'(0) = 0\}$,

with the norm

$$\|u\|_{W^2_2(R_+; H)}^2 = \left(\|u\|_{L_2(R_+; H)}^2 + \|A^2 u\|_{L_2(R_+; H)}^2\right).$$

Here and further the derivatives are considered in sense of distributions theory [1].

Consider the operator-differential equation

$$P(du/dt) = -u''(t) + \omega_1 A u(t) + \omega_2 A_0 u(t) = f(t), \quad t \in R_+ = (0, \infty) \tag{1}$$

with one of the initial boundary conditions

$$u(0) = 0 \tag{2}$$

or

$$u'(0) = 0 \tag{3}.$$  

Here $A$ is positively-defined self-adjoint operator, $A_0$ is linear, generally speaking, unbounded operator in $H$, the numbers $\omega_1 < 0$, $\omega_2 > 0$.

**Definition 1.** The problem $(1)$, $(2)/(1), (3)$ is called regularly solvable, if for each vector-function $f(t) \in L_2(R_+; H)$ there is unique vector-function $u(t) \in W^2_2(R_+; H)$, which satisfies the equation $(1)$ almost everywhere in $R_+$, and the boundary conditions $(2), (3)$ are fulfilled in sense of convergence of the norm of the space $H^{1/2}_{1/2}(H^{1/2}_{1/2})$, and the inequality...
\[ \|u\|_{W^1_2} \leq \text{const} \|f\|_{L^1_2} \]

takes place.

Sufficient solvability conditions of the problems (1), (2) and (1), (3) are shown in given paper. Note, that analogous problems for \( \omega_1 = -1, \ \omega_2 = 1 \) are studied, for example, in the works [2-4].

In proof of the main theorem on the regular solvability of the boundary value problems (1), (2) ((1), (3)) there are obtained the exact estimations of the norms of intermediate derivative \( A\frac{d}{dt} \) through the main part of the equation (1) in the spaces \( W^2_2(R, H; 0), W^2_2(R, H; 0) \) and \( W^2_2(R, H; 1) \). These estimations also have independent mathematical interest. First of all consider the equation

\[ P_1 \left( \frac{d}{dt} + \omega_1 A \right) u = \left( \frac{d}{dt} - \omega_1 A \right) \left( \frac{d}{dt} - w_2 A \right) u(t) = f(t), \quad t \in R, \]

with boundary condition (2) \( (\omega_1 < 0, \ \omega_2 > 0) \).

Denote by \( P_0 \) the operator, acting from the space \( W^2_2(R, H; 0) \) to \( L^2_2(R, H; 0) \) by the following way:

\[ P_0 u = P_0 \left( \frac{d}{dt} + A \right) u, \quad u \in W^2_2(R, H; 0). \]

It takes place

**Theorem 1.** Operator \( P_0 \) carries out the isomorphism from the space \( W^2_2(R, H) \) on the space \( L^2_2(R, H) \).

**Proof.** Obviously, the equation \( P_0 \varphi = 0 \) has only zero solution. From the other side, applying Plancherel theorem, it is easy to obtain that any \( f(t) \in L^2_2(R, H) \) vector-function

\[ u(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty P_0^{-1} (-i\xi, A) \left( \int_0^\infty f(\xi)e^{-i \xi t} d\xi \right) d\xi \]

belongs to the space \( W^2_2(R, H) \) and satisfies the equation (4) almost everywhere. Denote by \( \omega(t) \) narrowing \( u(t) \) on \([0, \infty) \) and we search the regular solution of the equation (4) in the form:

\[ u(t) = \omega(t) + e^{\omega(t)} \varphi, \quad \varphi \in H^1/2. \]

From the condition (2) we obtain that \( u(t) = \omega(t) - e^{\omega(t)} \varphi(0) \). As \( \omega(t) \in W^2_2(R, H) \), then according to the theorem on tracks [1, p.36] \( \omega(0) \in H^3/2 \).

Consequently, \( e^{\omega(t)} \varphi(0) \in W^2_2(R, H) \) and because of it \( u(t) \in W^2_2(R, H; 0) \).

From the other side according to the theorem on intermediate derivatives [1, p.29]

\[ \left\| P_1 u \right\|_{L^2_2} \leq 2 \left( \left\| u \right\|_{W^1_2}^2 + \left| \omega_1 \omega_2 \right| \left\| A^2 u \right\|_{L^2_2}^2 + \left| \omega_1 + \omega_2 \right| \left\| \frac{d^2}{dt^2} u \right\|_{L^2_2}^2 \right) \leq \text{const} \left\| u \right\|_{W^2_2}^2. \]

Consequently, according to Banach theorem on the inverse operator \( P_0 \) is the isomorphism between the spaces \( W^2_2(R, H; 0) \) and \( L^2_2(R, H; H) \). Theorem is proved.
Thus, the norms $\|u\|_{L^2_T(R ; H)}$ and $\|P_0u\|_{L^2_T(R ; H)}$ are equivalent in the space $W^1_2(R ; H : 0)$. That is why, according to the theorem on intermediate derivatives the number

$$N_1(R ; 0) = \sup_{0 \neq u \in W^1_2(R ; H : 0)} \left( \|Au\|_{L^2_T} \cdot \|P_0u\|_{L^2_T}^{-1} \right)$$

is finite.

It takes place

**Theorem 2.** Let $A$ be positively defined self-adjoint operator, the operator $A_1A^{-1}$ is bounded in $H$ and

$$\|A_1A^{-1}\| < N_1^{-1}(R , 0).$$

Then the problem (1')-(2) is regularly solvable.

**Proof.** We write the problem (1'), (2) in the form of operator equation

$$(P_0 + P_1)u = f,$$

where $P_0u = A_1u^*$, $f \in L_2(R ; H)$, $u \in W^1_2(R ; H : 0)$. According to theorem 1 operator $P_0^{-1}$ is bounded from $L_2(R ; H)$ to $W^1_2(R ; H : 0)$. After substitution $u = P_0^{-1} \phi$ we obtain the equation $(E + P_1P_0^{-1})\phi = f$ in $L_2(R ; H)$. As

$$\|P_1P_0^{-1}\|_{L_2} = \|P_1\|_{L_2} \leq \|A_1A^{-1}\| \|Au\| \leq N_1(R , 0) \|A_1A^{-1}\| \|P_0u\|_{L_2} = N_1(R , 0) \|A_1A^{-1}\| \|\phi\|_{L_2},$$

then if the inequality $N_1(R , 0) \|A_1A^{-1}\| < 1$ is fulfilled, the operator $E + P_1P_0^{-1}$ is invertible and we can find $u(t)$:

$$u(t) = \left( E + P_1P_0^{-1} \right)^{-1} f(t).$$

From here it follows that $\|u\|_{L^2_T} \leq \text{const} \|f\|_{L^2_T}$ (Theorem is proved).

Thus, for finding solvability conditions of the problem (1'), (2) we must find the exact value of $N_1(R , 0)$ or estimate it from above.

First of all we calculate the number

$$N_1(R , 0) = \sup_{\phi \neq 0} \left( \frac{\|Au\|_{L^2_T}}{\|P_0u\|_{L^2_T}^{-1}} \right).$$

(6)

Preliminarily we'll prove auxiliary statement.

**Lemma 1.** For any $u \in W^1_2(R ; H)$ and $\beta \in \left( 0 , (\omega_2 - \omega_1) \right)$ it takes place the identity

$$\|P_0^1 (d/dt : A) u \|_{L^2_T}^2 - \beta \|Au\|_{L^2_T}^2 = Q(\beta , \varphi_0 , \varphi_1) + \|\phi_1 (d/dt : \beta , A) u\|_{L^2_T}^2,$$

(7)

where $\varphi_0 = A^{1/2} u(0)$, $\varphi_1 = A^{1/2} u'(0)$.

$$\phi_1(x ; \beta , A) = \beta \beta A + \beta \beta A - \omega_2 \omega_2 A,$$

(8)

$$Q(\beta ; \varphi_0 , \varphi_1) = 4|\omega_2 \omega_2| \frac{\text{Re}(\varphi_1 , \varphi_0)}{\|\varphi_1\|^2} + \left( \frac{\|\omega_2 \omega_2\|^2 - \beta \beta \omega_2 \omega_2}{\|\varphi_1\|^2} \right)\|\varphi_1\|^2 +$$

$$+ |\omega_2 \omega_2| \left( \frac{\|\omega_2 \omega_2\|^2 - \beta \beta \omega_2 \omega_2}{\|\varphi_1\|^2} \right)\|\varphi_1\|^2.$$

(9)

**Proof.** Obviously, for all $u \in W^1_2(R ; H)$, the equality
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\[
\left\| P_0 \left( \frac{d}{dt} u \right) \right\|_{L_2}^2 = \left\| u^* \right\|_{L_2}^2 + \left| \omega_1 \omega_2 \right| \left\| A^2 u \right\|_{L_2}^2 + \left( \omega_1 + \omega_2 \right)^2 \left\| Au \right\|_{L_2}^2 + \\
2 \omega_1 \omega_2 \Re \left( u^*, A^2 u \right)_{L_2} - 2 \left( \omega_1 + \omega_2 \right) \Re \left( u^*, A^2 u \right)_{L_2} - 2 \omega_1 \omega_2 \left( \omega_1 + \omega_2 \right) \Re \left( Au, A^2 u \right)_{L_2}
\]

(10)
is true.

From the other side with integrating by part it is easy to verify the correctness of the following equalities

\[
\Re \left( u^*, A^2 u \right)_{L_2} = - \left\| Au \right\|_{L_2}^2 - \Re \left( \phi_1, \phi_0 \right), \\
2 \Re \left( u^*, A^2 u \right)_{L_2} = - \left\| \phi_0 \right\|_{L_2}^2.
\]

Taking into account the equalities (11) in (10) we obtain:

\[
\left\| P_0 \left( \frac{d}{dt} u \right) \right\|_{L_2}^2 = \left\| u^* \right\|_{L_2}^2 + \left| \omega_1 \omega_2 \right| \left\| A^2 u \right\|_{L_2}^2 + \left( \omega_1 + \omega_2 \right)^2 \left\| Au \right\|_{L_2}^2 - \\
\omega_1 \omega_2 \Re \left( \phi_1, \phi_0 \right) + \omega_1 \omega_2 \left( \omega_1 + \omega_2 \right) \left\| \phi_0 \right\|_{L_2}^2.
\]

As

\[
\phi_1 \left( \frac{d}{dt} : \beta : A \right) u \right\|_{L_2}^2 = \left\| u^* + \sqrt{\left( \omega_2 - \omega_1 \right)^2 - \beta} Au + \left( \omega_2 - \omega_1 \right) \left| Au \right|_{L_2}^2 - \\
\omega_1 \omega_2 \Re \left( Au, A^2 u \right)_{L_2}
\]

(12)
taking into consideration the equalities (11) and (12) in last identity we finish the lemma proof.

From this lemma we obtain the following

**Corollary 1.** The following identities take place:

a) for \( u \in W^2 \left( R_+ ; H : 0 : 1 \right) \) and \( \beta \in \left( 0, \left( \omega_2 - \omega_1 \right)^2 \right) \)

\[
\left\| P_0 u \right\|_{L_2}^2 - \beta \left\| Au \right\|_{L_2}^2 = \left\| \phi_1 \left( \frac{d}{dt} : \beta : A \right) u \right\|_{L_2}^2.
\]

(13)

b) for \( u \in W^2 \left( R_+ ; H : 0 \right) \) and \( \beta \in \left( 0, \left( \omega_2 - \omega_1 \right)^2 \right) \)

\[
\left\| P_0 u \right\|_{L_2}^2 - \beta \left\| Au \right\|_{L_2}^2 = \left\| \phi_1 \left( \frac{d}{dt} : \beta : A \right) u \right\|_{L_2}^2 + \left( \sqrt{\left( \omega_2 - \omega_1 \right)^2 - \beta} + \left( \omega_1 + \omega_2 \right) \right) \left\| \phi_0 \right\|_{L_2}^2.
\]

(14)

c) for \( u \in W^2 \left( R_+ ; H : 1 \right) \) and \( \beta \in \left( 0, \left( \omega_2 - \omega_1 \right)^2 \right) \)

\[
\left\| P_0 u \right\|_{L_2}^2 - \beta \left\| Au \right\|_{L_2}^2 = \left\| \phi_1 \left( \frac{d}{dt} : \beta : A \right) u \right\|_{L_2}^2 + \left( \omega_1 \omega_2 \right) \sqrt{\left( \omega_2 - \omega_1 \right)^2 - \beta} + \left( \omega_1 + \omega_2 \right) \left\| \phi_0 \right\|_{L_2}^2.
\]

(15)

Now we’ll find the exact value for \( N_1 \left( R_+ ; 0 : 1 \right) \).

**Theorem 3.** The number \( N_1 \left( R_+ ; 0 : 1 \right) = \left( \omega_2 - \omega_1 \right)^{-1} \), \( \omega_1 < 0, \omega_2 > 0 \).

**Proof.** In the equality (13) passing to the limit for \( \beta \to \left( \omega_2 - \omega_1 \right)^2 \), we obtain

\[
\left\| P_0 u \right\|_{L_2}^2 \geq \left( \omega_2 - \omega_1 \right) \left\| Au \right\|_{L_2}^2 
\]

for all \( u \in W^2 \left( R_+ ; H : 0 : 1 \right) \), i.e. \( N_1 \left( R_+ ; 0 : 1 \right) \leq \left( \omega_2 - \omega_1 \right)^{-1} \).

We’ll prove that \( N_1 \left( R_+ ; 0 : 1 \right) = \left( \omega_2 - \omega_1 \right)^{-1} \).

To do it for any \( \varepsilon > 0 \) it is sufficiently to construct the vector-function

\[
u \left( r \right) = \nu \left( r \right) \bigg|_{W^2 \left( R_+ ; H : 0 : 1 \right)}
\]

such that

\[
\left\| P_0 \nu \right\|_{L_2}^2 - \left( \left( \omega_2 - \omega_1 \right)^2 + \varepsilon \right) \left\| Au \right\|_{L_2}^2 < 0.
\]

(16)
We'll search $u_c(t) = g_c(t)\psi_c$, where $\psi_c \in H_4$, and $g_c(t)$ is scalar function from the class $W^2_2(R)$. For this function the inequality (16), according to Plancherel theorem, has the form

$$E(u_c) = \int q(\xi, \psi_c, A) \tilde{\psi}_c(\xi) d\xi \quad (\|\psi_c\| = 1),$$

(17)

where

$$q(\xi, \psi_c, A) = \left( p_0(-i, \xi, A)\tilde{p}_0(-i, \xi, A) - \left( (\omega_2 - \omega_0)^2 + \varepsilon \right) \xi^2 A^2 \right) \psi_c, \psi_c.$$  

(18)

Obviously, if the operator $A$ has eigen value $\mu$ and $A\psi_c = \mu\psi_c (\|\psi_c\| = 1)$, then minimum of the function on $\xi$

$$q(\xi, \psi_c) = p_0(-i, \xi, \mu)\tilde{p}_0(-i, \xi, \mu) - \left( (\omega_2 - \omega_0)^2 + \varepsilon \right) \xi^2 \mu^2$$

is negative for $\varepsilon > 0$.

If $A$ has not eigen vector, then $\mu > 0$ is its continuous spectrum and there is "almost eigen vector" $\psi_\varepsilon (\|\psi_\varepsilon\| = 1)$ such that $A\psi_\varepsilon = \mu\psi_\varepsilon + \delta$, where $\delta$ is the vector with sufficiently small norm. In this case minimum of the function $q(\xi, \psi_\varepsilon)$ is also negative. As the function $q(\xi, \psi_\varepsilon)$ is continuous on $\xi$, then there is an interval $(\eta_{\varepsilon}(\xi), \eta_{\varepsilon}(\xi))$ in which $q(\xi, \psi_\varepsilon) < 0$. Now let $\hat{\psi}_\varepsilon(\xi)$ is an arbitrary twice continuous function in $R$ with support in the interval $(\eta_{\varepsilon}(\xi), \eta_{\varepsilon}(\xi))$. If $g_c(t)$ is the inverse Fourier transformation of the function $\hat{\psi}_\varepsilon(\xi)$, then from the equality (17) it follows that

$$E(g_c(t)\psi_c) = \int_{\eta_{\varepsilon}(\xi)}^{\eta_{\varepsilon}(\xi)} q(\xi, \psi_c, \hat{\psi}_\varepsilon(\xi)) d\xi < 0.$$

As $E(\cdot)$ is continuous functional in the space $W^2_2(R, H)$, then from the theorem on density of the finite vector-functions in this space [1, p. 23] it follows that there is finite function $u_c(t) \in W^2_2(R, H)$ with support in $(-N, N) \subseteq R$, such that $E(u_c(t)) < 0$.

Supposing $u_c(t) = u_c(t - 2N)$ we obtain that $u_c(t) \in W^2_2(R, H; 0, 1)$ and $E(u_c(t)) < 0$.

Theorem is proved.

As $W^2_2(R, H; 0, 1) \subseteq W^2_2(R, H; 0)$, then it takes place

**Corollary 2.** The number $N_1(R, 0) \geq (\omega_2 - \omega_0)^{-1}$.

The case when $N_1(R, 0) = (\omega_2 - \omega_0)^{-1}$ is interesting.

It takes place

**Theorem 4.** The number $N_1(R, 0) = (\omega_2 - \omega_0)^{-1}$ if and only if $|\omega_2| \geq |\omega_0|$.  

**Proof.** Let $N_1(R, 0) = (\omega_2 - \omega_0)^{-1}$, then from the identity (14) it follows that for

$$\beta \in \left(0, (\omega_2 - \omega_0)^{-1}\right)$$

and $u \in W^1_2(R, H; 0)$ the inequality

$$\|\phi_1(\beta; A)u\|_{L^2}^2 + \left( \sqrt{\omega_2 - \omega_0} + \mu \right) \|\phi_0\|^2 =$$

$$= \left( p_0u \|_{L^2}^2 \left( 1 - \beta\right) \left( p_0u \|^2_{L^2} \right) \right) \|\phi_0\|^2 \left( \|1 - \beta N_1(R, 0)\|^2 \right) =$$

$$= \|p_0u \|^2 \left( 1 - \beta(\omega_2 - \omega_0)^2 \right) > 0.$$  

(19)

As both roots of characteristic equation $\phi_1(\lambda; \beta; \mu) = 0$...
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\[ \lambda_{1,2}(\beta; \mu) = \frac{1}{2} \left( \sqrt{(\omega_2 - \omega_1)^2 - \beta^2} - \sqrt{(\omega_2 + \omega_1)^2 - \beta^2} \right) \mu \]

\[ (\omega_2 > 0, \omega_1 < 0, \beta \in \{\omega_2 - \omega_1\}^2) \]

lie in the semi-plane, then Cauchy problem

\[ \phi_i(d/dt; \beta; A)u = 0, \quad u(0) = 0, \quad u'(0) = A^{1/2} \varphi_1, \quad \varphi_1 \in H \]

has solution \( u(t; \beta; \varphi_1) \in W^{1/2}_2(\Omega; H; 0). \)

Writing this solution in the inequality (19) we obtain that for any \( \beta \in \{0, (\omega_2 - \omega_1)^2\} \) the number \( \sqrt{(\omega_2 - \omega_1)^2 - \beta^2} + (\omega_1 + \omega_2) > 0 \).

Passing to the limit for \( \beta \to (\omega_2 - \omega_1)^2 \) we obtain \( |\omega_2| \geq |\omega_1| \).

Now let \( |\omega_2| \geq |\omega_1| \). Then the left part of the identity (14) is always positive and because of it for all \( \beta \in \{0, (\omega_2 - \omega_1)^2\} \) the inequality \( P_0 u_0 \|_{L_2^2}^2 \geq \beta \| A u' \|_{L_2^2}^2 \) is true. Passing to the limit for \( \beta \to (\omega_2 - \omega_1)^2 \) we obtain that \( N_i(\Omega; 0) \leq (\omega_2 - \omega_1)^{-1} \). Consequently, with taking into account the corollary 2 \( N_i(\Omega; 0) = (\omega_2 - \omega_1)^{-1} \). Theorem is proved.

**Theorem 5.** If \( |\omega_2| < |\omega_1| \), then \( N_i(\Omega; 0) = 2^{-1}|\omega_1\omega_2|^{-1/2} \).

**Proof.** Obviously, for \( |\omega_2| < |\omega_1| \) the number \( 4|\omega_1\omega_2| \in \{0, (\omega_2 - \omega_1)^2\} \). From theorem 4 it follows that \( N_i(\Omega; 0) > (\omega_2 - \omega_1)^{-1} \) (see corollary 1), i.e. \( N_i(\Omega; 0) \in \{0, (\omega_2 - \omega_1)^2\} \). If \( \beta \in \{0, N_i(\Omega; 0)\} \) for the solution of Cauchy problem (13) from the identity (14) it follows that

\[ \left( \sqrt{(\omega_2 - \omega_1)^2 - \beta^2} + (\omega_1 + \omega_2) \right) \| \varphi_1 \|_{L_2^2}^2 = \| P_0 u \|^2_{L_2^2} \| 1 - \beta \| A u' \|_{L_2^2}^2 \geq \| P_0 u \|^2_{L_2^2} \| 1 - \beta N_i^2(\Omega; 0) \|_{L_2^2}^2 > 0. \]

Consequently for \( \beta \in \{0, N_i(\Omega; 0)\} \) the function \( \theta(\beta) = \sqrt{(\omega_2 - \omega_1)^2 - \beta^2} + (\omega_1 + \omega_2) > 0 \).

From the other side for \( \beta \in \{N_i(\Omega; 0), (\omega_2 - \omega_1)^2\} \) according to definition of \( N_i(\Omega; 0) \) there is the vector-function \( \vartheta(t, \beta) \in W^2_2(\Omega; 0) \) such that \( \| P_0 \vartheta(t, \beta) \|_{L_2^2}^2 < \| A \vartheta(t, \beta) \|_{L_2^2}^2 \).

Because of it for \( \beta \in \{N_i(\Omega; 0), (\omega_2 - \omega_1)^2\} \) from the equality (14) it follows that

\[ \| \phi_i(d/dt; \beta; A) \vartheta(t, \beta) \|_{L_2^2}^2 + \left( \sqrt{(\omega_2 - \omega_1)^2 - \beta^2} + (\omega_1 + \omega_2) \right) \| \varphi_1 \|_{L_2^2}^2 < 0. \]

i.e. the function \( \theta(\beta) = \sqrt{(\omega_2 - \omega_1)^2 - \beta^2} + (\omega_1 + \omega_2) < 0 \) for \( \beta \in \{N_i(\Omega; 0), (\omega_2 - \omega_1)^2\} \).

As \( \theta(\beta) \) is continuous function of the argument \( \beta \), then \( \theta(N_i(\Omega; 0)) = 0 \), i.e. \( N_i(\Omega; 0) = 4|\omega_1\omega_2|^{-1/2} \). Consequently, \( N_i(\Omega; 0) = 2^{-1}|\omega_1\omega_2|^{-1/2} \). Theorem is proved.

**Corollary 3.** The number \( N_i(\Omega; 0) \) is defined by the following way

\[ N_i(\Omega; 0) = \begin{cases} (\omega_2 - \omega_1)^{-1} \text { for } |\omega_2| \geq |\omega_1|, \\ 2^{-1}|\omega_1\omega_2|^{-1/2} \text { for } |\omega_2| < |\omega_1|. \end{cases} \]
From the theorem 2 and corollary 3 it follows the following theorem on the solvability conditions of the problem (1), (2).

**Theorem 6.** Let $A$ be self-adjoint positively-defined operator, the operator $A_i A^{-1}$ is bounded in $H$ and the conditions

$$
\left\| A_i A^{-1} \right\| < \begin{cases} 
(\omega_2 - \omega_1) & \text{for } |\omega_2| \geq |\omega_1|,
\end{cases}
$$

$$
(2|\omega_1\omega_2|)^{1/2} & \text{for } |\omega_2| < |\omega_1|,
\end{cases}
$$

are satisfied. Then the problem (1), (2) is regularly solvable.

From this theorem it follows that the class of operators $A_i$ satisfying the solvability conditions of the problem (1), (2) is wide, if $|\omega_2| > |\omega_1|$. 

Using the equality (15) we can analogously prove the following

**Theorem 7.** The number

$$
N_i(R_i, \lambda) = \sup_{\lambda \in \Omega_i} \left\{ \left\| Au \right\|_{L_2^2} \cdot \left\| P_i u \right\|_{L_2^2}^{-1} \right\}
$$

is defined by the following way

$$
N_i(R_i, \lambda) = \begin{cases} 
(\omega_2 - \omega_1)^{-1} & \text{for } |\omega_2| \leq |\omega_1|, \\
2^{-1/2} |\omega_1\omega_2|^{-1/2} & \text{for } |\omega_2| > |\omega_1|.
\end{cases}
$$

**Theorem 8.** Let $A$ be self-adjoint positively defined operator, the operator $A_i A^{-1}$ is bounded in $H$ and the conditions

$$
\left\| A_i A^{-1} \right\| < \begin{cases} 
(\omega_2 - \omega_1) & \text{for } |\omega_2| \leq |\omega_1|,
\end{cases}
$$

$$
(2|\omega_1\omega_2|)^{1/2} & \text{for } |\omega_2| > |\omega_1|,
\end{cases}
$$

are satisfied. Then the problem (1), (3) is regularly solvable.

From this theorem it follows that the class of the operators $A_i$ satisfying solvability conditions of the problem (1), (3) is wide, if $|\omega_2| < |\omega_1|$. 

References


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